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Comment on "Investment Timing under Incomplete Information"

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In a recent contribution to this journal, Décamps, Mariotti, and Villeneuve (2005) analyze the decision of when to invest in a project whose value is perfectly observable but driven by a parameter that is unknown to the decision maker ex ante. Using filtering and martingale techniques, they find that (i) the decision maker always benefits from an uncertain drift relative to an average drift situation, and (ii) drift uncertainty unambiguously delays investment. Using the principle of smooth fit, I derive an analytical solution to the problem and give a numerical example that shows that both claims do not hold true in general. My analysis shows that the impact of drift uncertainty on the value of the option to invest and the optimal timing of investment is governed by two separate effects: the impact of uncertainty per se and the impact of learning. In particular, the authors' results only hold true if the latter outweighs the former.

Key words: investment under uncertainty; optimal stopping; free boundary; filtering

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1. Introduction. The purpose of this comment is twofold. First, I derive an analytical solution to the optimal stopping time problem $\mathcal{P}$ considered in [2]. Hence, I find a closed-form expression for the value of the option to invest defined by

$$G^*(V_0, P_0) = \sup_\tau \mathbb{E}\left[e^{-r\tau}V_\tau - I\right], \tag{1}$$

where the current payoff of the investment opportunity V_t and the decision maker's beliefs $P_t = \Pr\left[\mu = \mu_h \middle| \mathcal{F}_t\right]$ about the drift of V_t follow, respectively,

$$\begin{bmatrix} dV_t/V_t \\ dP_t \end{bmatrix} = \begin{bmatrix} \mu_l + P_t(\mu_h - \mu_l) \\ 0 \end{bmatrix} dt + \begin{bmatrix} \sigma \\ \omega P_t(1-P_t) \end{bmatrix} d\bar{W}_t, \tag{2}$$

and $\omega \equiv (\mu_h - \mu_h)/\sigma$ represents the signal-to-noise ratio of the updating process. The analytical solution obtains whenever $\mu_l + \mu_h = \sigma^2$. This is so, since the latter condition establishes a one-to-one mapping between payoff V_t and beliefs P_t, which, in turn, reduces $\mathcal{P}$ to a one-dimensional optimal stopping time problem. In this case, the optimal stopping time is a trigger strategy, that is, $\tau^* = \inf\{t : V_t \geq V^*\}$, where the optimal investment threshold V^* is a constant that can be determined by the principle of smooth fit.[1]

Second, on the basis of the analytical expression for $G^*(V_0, P_0)$, I give a numerical counterexample to the claim of Theorem 7.2 in [2]: I thereby show that the value of the option to invest need not increase with the introduction of drift uncertainty.[2] Correspondingly, a mean-preserving spread around some average drift need not delay investment. Rather, the impact of drift uncertainty on option value $G^*(V_0, P_0)$ and optimal investment threshold V^* can be decomposed into the impact of uncertainty per se and the (separate) impact of the decision maker's learning process. In particular, I show that Theorem 7.2 in [2] only holds true if the latter outweighs the former. A purely numerical analysis of problem $\mathcal{P}$ that I present in [4] for the case in which $\mu_l + \mu_h \neq \sigma^2$ strongly suggests that this finding is robust.[3]

[1] Since I reduce $\mathcal{P}$ to a *one-dimensional* optimal stopping time problem before applying the principle of smooth fit, the concerns that [2] voice with respect to the applicability of the latter do not apply. See page 473 of the original contribution for a discussion of the issues involved when considering the *bivariate* problem.

[2] Recall that [2] consider the special case in which $\mu_l = 0$ and $\mu_h = 1$. My analysis accomodates the original contribution by considering the case in which $\mu_l + \mu_h = \sigma^2 = 1$.

[3] There, I consider the partial differential equation that corresponds to $\mathcal{P}$ by the Feynman-Kac theorem. Since outside

2. Analytical Solution of the Problem. To solve (1) explicitly, I perform a change of variable and define a new value function:

LEMMA 2.1 *Consider the likelihood ratio $\phi_t = P_t/(1-P_t)$ with initial value $\phi_0 = P_0/(1-P_0)$ and the value function defined by*

$$\Sigma^*(V_0, \phi_0) \equiv (1+\phi_0)\, G^*\left(V_0, \frac{\phi_0}{1+\phi_0}\right). \tag{3}$$

From (1), Girsanov's theorem implies

$$\Sigma^*(V_0,\phi_0) = \sup_\tau \mathbb{E}_{\mu_l}\left[e^{-r\tau}(1+\phi_\tau)(V_\tau - I)\right], \tag{4}$$

where $\mathbb{E}_{\mu_l}$ denotes the expectation operator with respect to the laws of motion

$$\begin{bmatrix} dV_t/V_t \\ d\phi_t/\phi_t \end{bmatrix} = \begin{bmatrix} \mu_l \\ 0 \end{bmatrix} dt + \begin{bmatrix} \sigma \\ \omega \end{bmatrix} dW_t, \tag{5}$$

and $dW_t = \omega\phi_t/(1+\phi_t)\,dt + d\bar{W}_t$.

PROOF. Consider the process $\lambda_t = (\hat{\mu}_t - \mu_l)/\sigma$, where $\hat{\mu}_t = \mu_l + P_t(\mu_h - \mu_l)$ is the expected growth rate of V_t as of time t. Then, $\lambda_t = \omega P_t = \omega\phi_t/(1+\phi_t)$, and we have the exponential Martingale

$$\eta_t = \exp\left(\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds\right),$$

where $dW_s = \lambda_s ds + d\bar{W}_s$. By Girsanov's theorem[4], we obtain the Radon-Nikodym derivative $\eta_t = dP/dP_{\mu_l}$ such that

$$\sup_\tau \mathbb{E}\left[e^{-r\tau}(V_\tau - I)\right] = \sup_\tau \mathbb{E}_{\mu_l}\left[e^{-r\tau}\eta_\tau(V_\tau - I)\right].$$

It is left to show that $\eta_\tau = (1+\phi_\tau)/(1+\phi_0)$. Clearly, $\eta_0 = 1$. Moreover, Itô's Lemma and the law of motion for ϕ_t from (2) imply $d\phi_t/\phi_t = \omega dW_t + \omega^2 P_t dt$, which, in turn, implies $d\phi_t/(1+\phi_t) = \lambda_t dW_t = d\eta_t/\eta_t$. □

Lemma 2.1 disentangles the dynamics of the payoff of the investment opportunity from the dynamics of learning: by changing the drift of V_t from $\hat{\mu}_t$ to μ_l, I represent the optimal stopping time problem under incomplete information (for $G^*(V_0, P_0)$ the drift of V_t is unknown and updated continuously) as an optimal stopping time problem under complete information (for $\Sigma^*(V_0,\phi_0)$ the drift of V_t is known and constant). Since it coincides with the Radon-Nikodym derivative that defines a change of drift from μ_l to $\hat{\mu}_t$, the multiplicative payoff adjustment $(1+\phi_\tau)/(1+\phi_0)$ for $\tau \geq 0$ ensures that both formulations are equivalent.

Next, defining the constant $\varepsilon \equiv (\mu_l + \mu_h - \sigma^2)/2$ and integrating (5), we immediately have $(\phi_t/\phi_0)^\sigma = e^{\varepsilon\omega t}(V_t/V_0)^\omega$. The last equation, in turn, establishes a direct link between the dynamics of learning and the current payoff of the investment opportunity. In particular, it shows that time t and the current position of V_t relative to some initial value V_0 are a sufficient statistic for ϕ_t relative to some initial belief ϕ_0. Moreover, it is immediately obvious that for $\mu_l + \mu_h = \sigma^2$ the relationship between V_t and ϕ_t is independent of time: when $\varepsilon = 0$, then, for some initial values V_0 and ϕ_0, we obtain the one-to-one mapping

$$\phi_t = \phi_0\left(\frac{V_t}{V_0}\right)^{\frac{\omega}{\sigma}}. \tag{6}$$

It is then possible to reduce the dimensionality of optimal stopping time problem $\mathcal{P}$ to one state variable by substituting (6) into (4). This, in turn, allows for an analytical solution of (4), and thereby (1):

PROPOSITION 2.1 (*i*) *For $\mu_l + \mu_h = \sigma^2$, the optimal stopping time that solves (1) is a trigger strategy with $\tau^* = \inf\{t : V_t \geq V^*\}$. The constant investment threshold V^* depends on initial values V_0 and P_0 and is uniquely defined by*

$$V^* = \frac{f_l + \frac{P_0}{1-P_0}\left(\frac{V^*}{V_0}\right)^{\frac{\omega}{\sigma}} f_h}{f_l - 1 + \frac{P_0}{1-P_0}\left(\frac{V^*}{V_0}\right)^{\frac{\omega}{\sigma}}(f_h - 1)} I, \tag{7}$$

the parameter restriction the problem is isomorphic to the valuation of an American put option with finite maturity, I resort to an implicit finite-difference scheme that solves the free boundary problem in linear complementarity form.

[4] It is easy to verify that the Novikov condition holds true.

where f_i follows, for $\mu_i \in \{\mu_l, \mu_h\}$, from

$$f_i = -\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{8}$$

(*ii*) *The value of the option to invest is*

$$G^*(V_t, P_t) = (1 - P_t)\,\hat{G}_l(V_t; V^*) + P_t \hat{G}_h(V_t; V^*), \tag{9}$$

where $\hat{G}_i(V_t; V^*)$ follows, for $f_i \in \{f_l, f_h\}$, from

$$\hat{G}_i(V_t; V^*) = \left(\frac{V_t}{V^*}\right)^{f_i}(V^* - I).$$

Proof. From (4) and (6), we have

$$\Sigma^*(V_t, \phi_t) = \gamma(V_t) = \sup_\tau \mathbb{E}_{\mu_l}\left[e^{-r\tau}\left(1 + \phi_0\left(\frac{V_\tau}{V_0}\right)^{\frac{\omega}{\sigma}}\right)(V_\tau - I)\right].$$

The last equation defines a standard one-dimensional optimal stopping time problem: since V_t follows a canonical diffusion process with constant drift μ_l and volatility σ, the conditions of the Feynman-Kac theorem are satisfied. Hence, from [5] we know that $\gamma(V_t)$ satisfies the variational inequality

$$\max\left[\mathcal{L}\gamma(V_t) - r\gamma(V_t),\ \left(1 + \phi_0\left(\frac{V_t}{V_0}\right)^{\frac{\omega}{\sigma}}\right)(V_t - I) - \gamma(V_t)\right] = 0, \tag{10}$$

where $\mathcal{L}$ denotes the differential operator. Without loss of generality, we can then consider the optimal stopping time to be of the first-passage type, that is, $\tau = \inf\{t : V_t \geq V_I\}$, where the optimal choice of V_I is still to be determined.[5] From (10), inside the continuation region (that is, for $V_t < V_I$), $\gamma(V_t)$ therefore needs to satisfy the homogeneous Cauchy-Euler ordinary differential equation

$$\frac{1}{2}\sigma^2 V_t^2 \gamma''(V_t) + \mu_l V_t \gamma'(V_t) - r\gamma(V_t) = 0,$$

subject to the boundary condition $\gamma(0) = 0$ and the value matching condition $\gamma(V_I) = (1 + \phi_0 (V_I/V_0)^{\frac{\omega}{\sigma}})(V_I - I)$. From this, we immediately obtain

$$\gamma(V_t) = \left(\frac{V_t}{V_I}\right)^{f_l}\left(1 + \phi_0\left(\frac{V_I}{V_0}\right)^{\frac{\omega}{\sigma}}\right)(V_I - I). \tag{11}$$

Next, as [1] and [5] show, for the simple problem under consideration, the optimal choice of investment threshold V_I, denoted by V^*, is obtained by imposing the smooth pasting condition[6]

$$\left.\frac{\partial}{\partial V_t}\gamma(V_t)\right|_{V_t = V^*} = \left.\frac{\partial}{\partial V_t}\left(\left(1 + \phi_0\left(\frac{V_t}{V_0}\right)^{\frac{\omega}{\sigma}}\right)(V_t - I)\right)\right|_{V_t = V^*}.$$

However, noting that $\gamma(V_t)$, as defined by (11), is globally concave in V_I, the optimal investment threshold V^* can just as well be determined from setting the first derivative of (11) with respect to V_I equal to zero. It is easy (albeit tedious) to verify that both procedures lead to the following equation that uniquely determines V^*:

$$V^* = \frac{f_l + \phi_0\left(\frac{V^*}{V_0}\right)^{\frac{\omega}{\sigma}}\left(f_l - \frac{\omega}{\sigma}\right)}{f_l - 1 + \phi_0\left(\frac{V^*}{V_0}\right)^{\frac{\omega}{\sigma}}\left(f_l - \frac{\omega}{\sigma} - 1\right)} I.$$

Since $\phi_0 = P_0/(1 + P_0)$ and, for $\mu_l + \mu_h = \sigma^2$, we have $f_l - \frac{\omega}{\sigma} = f_h$, we then obtain (7).

[5] See [3], [6], and in particular [1] for a formal proof of the optimality of a trigger strategy.
[6] Assumptions 1-3 in [5] are trivially satisfied such that their Theorem 8 applies.

To prove the second part of Proposition 2.1, recall from (3) that $(1+\phi_t)\, G^*\left(V_t, \phi_t/(1+\phi_t)\right) = \Gamma^*(V_t, \phi_t) = \gamma(V_t)$. Hence, we get

$$\begin{aligned}(1+\phi_t)\, G^*\left(V_t, \frac{\phi_t}{1+\phi_t}\right) &= \left(\frac{V_t}{V^*}\right)^{f_l}\left(1+\phi_0\left(\frac{V^*}{V_0}\right)^{\frac{\omega}{\sigma}}\right)(V^*-I)\\ &= \left(\left(\frac{V_t}{V^*}\right)^{f_l} + \phi_0\left(\frac{V_t}{V_0}\right)^{\frac{\omega}{\sigma}}\left(\frac{V_t}{V^*}\right)^{f_l-\frac{\omega}{\sigma}}\right)(V^*-I).\end{aligned}$$

From (6) and $f_l - \frac{\omega}{\sigma} = f_h$, we finally obtain

$$G\left(V_t, \frac{\phi_t}{1+\phi_t}\right) = \frac{\left(\left(\frac{V_t}{V^*}\right)^{f_l} + \phi_t\left(\frac{V_t}{V^*}\right)^{f_h}\right)(V^*-I)}{1+\phi_t},$$

which, upon substitution of $\phi_t = P_t/(1-P_t)$, proves (9). □

The first part of Proposition 2.1 shows that when the relationship between V_t and ϕ_t (and thereby the relationship between V_t and P_t) is independent of time – as captured by (6) – then the optimal stopping time that solves (1) can be represented as a trigger strategy. The second part of Proposition 2.1 then interprets the value of the option to invest under incomplete information as a belief-weighted average of the complete information benchmarks that obtain for the low and the high drift, taking the investment threshold V^* from (7) as a given.[7]

Next, inspection of (7) in conjunction with (6) reveals that the optimal investment threshold V^* can be interpreted as a harmonic average of the respective complete information benchmarks $f_i/(f_i-1)\, I$ that obtain for $\mu_i \in \{\mu_l, \mu_h\}$ with $f_i \in \{f_l, f_h\}$: we have

$$V^* = \frac{f_l + \phi^* f_h}{f_l - 1 + \phi^*(f_h-1)} I = \frac{(1-P^*)\, f_l + P^* f_h}{(1-P^*)(f_l-1) + P^*(f_h-1)} I,$$

where ϕ^* and P^* denote the decision maker's beliefs at investment.[8] Congruent with this interpretation, we also have the following result:

Corollary 2.1 *The belief-dependent investment boundary $v^*(P)$, which from Theorem 7.1 in [2] is non-decreasing and continuous on* $[0,1]$, *is given by*

$$v^*(P) = \frac{(1-P)\, f_l + P f_h}{(1-P)(f_l-1) + P(f_h-1)} I. \qquad (12)$$

Proof. The result immediately follows from solving the optimal stopping time problem for the belief-weighted average given by equation (9). □

Note that the constant investment threshold V^* from (7) is dynamically consistent with moving boundary (12) that defines the investment region: this is so, since, for as long as the parameter restriction $\mu_l + \mu_h = \sigma^2$ applies, the law of motion of V_t, combined with (12) and the one-to-one mapping

$$P_t = \frac{1}{1 + \frac{1-P_0}{P_0}\left(\frac{V_0}{V_t}\right)^{\frac{\omega}{\sigma}}}, \qquad (13)$$

allows the decision maker to perfectly anticipate the first time that V_t hits $v^*(P_t)$ from below.[9] In V_t and P_t-dimension, these are the fixed points V^* and P^*: $V^* = V_{\tau^*} = v^*(P_{\tau^*}) = v^*(P^*)$, where $\tau^* = \inf\{t : V_t \geq V^*\}$ and P^* follows from (13) with $V_t = V^*$. Hence, from (7) and (13), the optimal stopping time τ^* could equally well be characterized as $\tau^* = \inf\{t : P_t \geq P^*\}$. In summary, inside the continuation region we have $V_t \leq V^*$ and $P_t \leq P^*$ for all $t \leq \tau^*$. The decision maker optimally invests the first time that $V_t = V^*$ (which implies $P_t = P^*$, and vice versa) with $V^* = v^*(P^*)$.

[7] Note that V^* would not be optimal if the decision maker knew for sure that the drift is either low or high.
[8] Beliefs at investment P^* are further characterized below.
[9] Note that (13) is immediately obvious from $P_t = \phi_t/(1+\phi_t)$ and (6).

Insert Figure 1 here.

Figure 1 illustrates the result: since there is a one-to-one mapping between V_t and P_t, the horizontal axis can be interpreted both in terms of the current payoff of the investment or beliefs. Correspondingly, moving boundary (12) is represented by the convex curve that is bounded by the complete information benchmarks $v^*(0) = f_l/(f_l - 1) I$ and $v^*(1) = f_h/(f_h - 1) I$. $v^*(P)$ slopes upwards, as more optimistic beliefs about the true drift of the payoff process favor later investment. For given intitial values V_0 and P_0 we thus obtain the initial threshold $v^*(P_0)$. On the other hand, the intersection of (12) with the 45-degree line determines the optimal investment threshold V^*. A final, intuitively appealing interpretation of the optimal investment threshold proceeds as follows:

COROLLARY 2.2 *Define* $v_i(V_t) \equiv f_i I - (f_i - 1) V_t$ *for* $f_i \in \{f_l, f_h\}$ *as the value of continuation in the respective benchmark case of complete information. Then, the optimal investment threshold and beliefs at investment satisfy* $(1 - P^*) v_l(V^*) + P^* v_h(V^*) = 0$, *where* V^* *and* P^* *are given by (7) and (13) with* $V_t = V^*$.

PROOF. From before. □

Under complete information, we have $v_i(V_t) \geq 0$ whenever $V_t \leq f_i/(f_i - 1) I$: continuation is optimal when the true growth rate is known to be either $\mu_i \in \{\mu_l, \mu_h\}$ and V_t is smaller than the respectively optimal investment threshold. With drift uncertainty, however, investment occurs when the belief-weighted average of the value of continuation vanishes: it is precisely then that the risk of investing too late $(v_l(V^*) < 0)$ balances the risk of investing too early $(v_h(V^*) > 0)$.

3. Numerical Counterexample to Theorem 7.2. Recall that [2] compare the investment problem under incomplete information with an average drift scenario in which $\hat{\mu} = \mu_l + P_0(\mu_h - \mu_l)$ is used in the analytical expressions that obtain in the benchmark case of complete information. Figure 1 is drawn in such a way that the optimal investment threshold for an average drift, as represented by $\hat{V} = f(\hat{\mu}) / (f(\hat{\mu}) - 1) I$, lies above the optimal threshold under drift uncertainty, as represented by V^*. As I show in the following, this need not be the case but is a possibility that is ignored in [2].

In accordance with [2], I presume $\mu_l = 0$, $\mu_h = 1$, and, so as to match the parameter restriction, set $\sigma^2 = 1$. With $P_0 = 0.5$, I thus consider a mean-preserving spread around $\hat{\mu} = 0.5$. The remaining parameters are two distinct initial values $V_0^1 = 100$ and $V_0^2 = 200$, as well as the interest rate $r = 1.5$, and the cost of investment $I = 100$. Panel 1 shows the results that obtain for the initial position of the belief-dependent investment boundary $v^*(P_0)$ from (12), the optimal investment threshold V^* from (7), beliefs at investment P^* from (13) with $V_t = V^*$, the value of the option to invest $G^*(V_0, P_0)$ from (9), and the price of the Arrow-Debreu security $\mathbb{E}\left[e^{-r\tau^*}\right]$:[10]

μ_l	μ_h	V_0	$v^*(P_0)$	V^*	P^*	$G^*(V_0, P_0)$	$\mathbb{E}\left[e^{-r\tau^*}\right]$
0.5	0.5	100	236.60	236.60	0.5000	30.73	0.2249
0.0	1.0	100	224.56	275.78	0.7338	31.94	0.1817
0.5	0.5	200	236.60	236.60	0.5000	102.10	0.7474
0.0	1.0	200	224.56	230.27	0.5351	101.29	0.7775

Panel 1

From Panel 1 we see that, for both $V_0^1 = 100$ and $V_0^2 = 200$, drift uncertainty has two effects: first, for given beliefs P_0, the optimal investment threshold $v^*(P_0) = 224.56$ is always lower than $\hat{V} = f(\hat{\mu}) / (f(\hat{\mu}) - 1) I = 236.60$. This is so, since, from (8), $f(\cdot)$ is convex and $f(\cdot) > 0$. As [2] argue on page 483, the result illustrates "an implicit risk aversion due to the additional uncertainty generated by the randomness of μ". Put differently, for given beliefs P_0, the introduction of uncertainty per se

[10] It is well-known that, under complete information, the price of the Arrow-Debreu security corresponds to $(V_t/V_I)^{f_i}$ for $f_i \in \{f_l, f_h\}$. Under incomplete information it is easily calculated as a belief-weighted average of the complete information benchmarks.

accelerates investment. At the same time, the dynamics of learning *delay* investment: since (i) from (13) a positive innovation in V_t correlates positively with P_t, and (ii) from (12) more optimistic beliefs favor later investment ($v^{*\prime}(P) > 0$), an increase in V_t raises the belief-dependent investment threshold $v^*(P_t)$. Whether it is the first or the second effect that dictates the overall impact of drift uncertainty on V^* therefore depends on parameter values. Comparing the first with the second row, we see that when $V_0^1 = 100$ is comparatively distant from the initial investment threshold $v^*(P_0) = 224.56$, then, by the time that V_t has reached a level close to $v^*(P_t)$, the decision maker's beliefs P_t are sufficiently optimistic such that $V^* > \hat{V}$. On the other hand, comparing the third with the fourth row, in the case in which $V_0^2 = 200$ is comparatively close to the initial investment threshold $v^*(P_0) = 224.56$, then there is little scope for learning. Correspondingly, it is the impact of uncertainty per se that governs the optimal investment threshold and $V^* < \hat{V}$.

To gain a better understanding of the effects at work, in Panel 2 I isolate the impact of uncertainty by keeping the investment threshold V_I fixed at the level $\hat{V} = 236.60$ that is optimal for the average drift scenario:

μ_l	μ_h	V_0	$v^*(P_0)$	V_I	P_I	$G(V_0, P_0)$	$\mathbb{E}[e^{-r\tau}]$
0.5	0.5	100	236.60	236.60	0.5000	30.73	0.2249
0.0	1.0	100	224.56	236.60	0.7029	31.64	0.2316
0.5	0.5	200	236.60	236.60	0.5000	102.10	0.7474
0.0	1.0	200	224.56	236.60	0.5419	101.25	0.7412

Panel 2

We clearly see that, as in Panel 1, drift uncertainty has an *asymmetric* impact on both the value of the option to invest, as captured by $G(V_0, P_0)$, and the likelihood of investment, as captured by $\mathbb{E}[e^{-r\tau}]$, even when keeping the investment threshold fixed at a suboptimal level. For $V_0^1 = 100$ it is the upside potential of drift uncertainty that raises both quantities, whereas for $V_0^2 = 200$ it is the downside risk that decreases the value of the option to invest and the likelihood of investment.

4. Conclusion. I have shown that the impact of drift uncertainty on both the value of the option to invest and the optimal timing of investment is governed by two separate effects: the impact of uncertainty per se and the impact of learning. Since the relative magnitude of both effects depends on parameter values, the overall impact is ambiguous. I have given a numerical example for both cases: in the first scenario Theorem 7.2 in [2] holds true and drift uncertainty raises the value of the option to invest, thereby delaying investment; in the second scenario Theorem 7.2 in [2] does not hold true and drift uncertainty reduces the value of the option to invest, thereby accelerating investment.

The results follow a simple economic logic: whether or not the decision maker is willing to accept a gamble on the drift of the investment project depends on how close to being exercised the option to invest is for an average drift. If the option is comparatively "far out of the money", then the upside potential of a mean-preserving spread outweighs the downside risk. If, however, the option is comparatively "close to the money", then the downside risk outweighs the upside potential. Intuitively, in the latter case the drift of the investment project is of second-order importance: already for an average drift the realization of the investment project's payoff, the magnitude of which is drift-independent, is imminent. It is precisely then that "a bird in the hand is worth two in the bush".

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Figure 1